THE

MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.; E. T. WHITTAKER, M.A., F.R.S.; W. E. HARTLEY, B.A.

LONDON:

GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN, AND BOMBAY.

VOL. III. JANUARY, 1906. No. 55.

THE TEACHING OF MECHANICS.

In its revised form the Report of the M.A. Committee on the Teaching of Elementary Mechanics strikes me as being on the whole an exceedingly good and helpful one, though of course open to criticism in some of its details. On two or three points relating to Kinetics, I offer these remarks:

1.

Average velocity during a given interval would be better defined as 'that imagined uniform velocity with which the actual displacement would be accomplished during the given interval.' This of course implies that we have already defined *uniform velocity* (preferably as that of a body moving so that the displacement is proportional to the time taken). Then the formula: Measure of velocity = measure of displacement ÷ measure of time-interval, having been proved for *uniform velocity*, would apply at once to *average velocity* in motion of any kind.

The notion of the *average* as being that which, substituted throughout for the variable, would give the *same result*, is perhaps the fundamental one, and is certainly one which can be clearly grasped by the mathematical tyro. The average height of seven mountains, *e.g.* is the height of one of seven equally high mountains which would give the same total height when piled on the top of one another as would the seven unequal mountains. Of course the word *result* in the general statement is vague: but the appropriate meaning is obvious in most common cases of *average*.

I would also suggest that the distinction between the terms *velocity* and *speed* ought to be explained, and that in connection with the former we ought to speak of *displacements accomplished* rather than of *distances described*.

2.

Centrifugal force will never be abandoned by practical dynamists; so we had better keep it in the theory. It is as good as the term *sunrise*, which is equally incorrect from one point of view. Theoretical dynamists had better explain 'centrifugal force' rather than reject it. And the explanation can be made quite clear if we treat it as a particular case of 'reversed effective force' (to use an old phrase). If a body of mass m lbs. has an acceleration of a feet per second per second, the resultant force on it must be $m \times a$ poundals; *i.e.*, it would be in equilibrium if a force of $-ma$ poundals acted on it in addition to the actual forces. This imaginary force then, along with the actual forces, will satisfy the conditions of an imaginary equilibrium. This mode of explanation removes somewhat of the paradoxical character of 'centrifugal force' by classing it with like things, and also prepares the way for a subsequent understanding of 'D'Alembert's Principle.' The importance of the 'reversed effective force' is that, though imaginary, it is equal in all respects to the real reaction of the force that 'balances' it in the imaginary equilibrium; for example, in the case of the stone swung round at the end of a string, under no external forces, it is equal to the pull which the stone exerts on the string; or in the case of the piston near the end of its stroke, the extra stress in the piston rod due to negative acceleration of the piston rod is exactly equal to the 'reversed effective force on the piston.' Such 'inertia reactions' are becoming more important to engineers as the speeds of engines increase, as can be seen by perusing recent technical publications on vibrations and balancing of engines. And it seems to me that school mechanics ought to give a basis for understanding the language and ideas of those who apply the science, as well as an introduction to dynamics as a pure science.

3.

Impulse is not referred to in the Report. I should have liked to see a recommendation that the *impulse of a force during a given interval* should be defined as *the change of momentum it produces*, rather than (as in some books) *the product of Force and interval of time*, since the former definition, unlike the latter, applies at once whether the force be uniform or not. Besides, the former definition seems to suggest the fundamental idea better. In connection with the *Third Law of Motion*, the term *Stress* ought to be introduced, and defined after the manner of Clerk Maxwell to be the totality of that phenomenon of which the *Action* and the *Reaction* are one-sided aspects. The fact that every force implies the existence of a body acting, as well as a body acted upon, might be emphasized here.

R. F. MUIRHEAD.

ON A BINOMIAL APPROXIMATION.

THE simple approximation for the rth root of a number which is nearly an exact rth power, given by Mr. Whipple last month, is found as a rule in Bonnycastle's *Arithmetic*. Perhaps the most elegant form is this:

If a and b are nearly equal, then the nth root of a/b is approximately

$$\frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}. \quad \text{.......................}(1)$$

This is of course the same as

$$(1+x)^{\frac{1}{n}}=\frac{2n+(n+1)x}{2n+(n-1)x}, \text{ nearly. } \quad \text{.................}(2)$$

For example, let $a=2$, $b=1$, $n=3$, then a first approximation to the cube root of 2 is 1·25. Making this the basis for a second, we have

$$2^{\frac{1}{3}} \doteq \frac{5}{4}\cdot\left(\frac{128}{125}\right)^{\frac{1}{3}},$$

and applying the rule to 128/125 we get at once the approximation 635/504 or 1·2599206 ..., which is correct to 6 decimal places.

By ordinary expansion of both sides in (2), we find the difference to be $x^3(n^2-1)/12n^3$. In our second case $x=3/125$, $n=3$, and the error is nearly ·00000042.

It can be proved by a fairly long induction, or otherwise, that this result is a particular case of the more general theorem

$$(1+x)^n=\left\{1+\frac{(n+r)r}{2r}x+\frac{(n+r)(n+r-1)r(r-1)}{2r(2r-1)1\,.\,2}x^2+\dots\right\}$$
$$\div\left\{1-\frac{(n-r)r}{2r}x+\frac{(n-r)(n-r+1)r(r-1)}{2r(2r-1)1\,.\,2}x^2-\dots\right\},$$

where the factor $r-r$ in the two $(r+2)$th terms causes evanescence of the numerators until at the $(2r+2)$th term the denominators also have the vanishing factor $(2r-2r)$ and the terms are restored.

This is an exact theorem of which a mere general case will be found in his *Methodus nova integralium valores per approximationem inveniendi*, §§ 40–42.

This note appeared, in substance, in the *Proceedings of the Edinburgh Mathematical Society*, vol. x., p. 23.

J. E. A. STEGGALL.

JAPANESE MATHEMATICS.

THE *Jahresbericht der Deutschen Mathematiker-Vereinigung* has lately published an address given by Professor Harzer on "The Exact Sciences in Old Japan." At first sight it seems difficult to see why the celebration of the birthday of the Emperor of the Germans should have been selected for the delivery of an oration on this topic. The Professor is himself an astronomer, so that it is interesting to note that he brushes aside as of little value anything that Japan has accomplished in the pursuit of his favourite science. Nor does he discover any ground for supposing that anything will be accomplished along this line by those whose sudden prominence in other realms has given the Western world so furiously to think. The history of the science of mathematics in the Island Empire is full of paradoxes. In the days of Pythagoras, it will be remembered, the secrets of the science were entrusted to what was to all intents and purposes a Masonic body, bound by oath not to divulge the mysteries in which they participated. And this body had a further curious and unexpected resemblance to the Masonic order as seen among the Latin races, inasmuch as its intrusion into politics led to more than one popular outbreak. In Japan the parallel almost holds good, save for the utilisation of the society for political purposes. The elementary parts of the science were open to all. He who ran might read. But before the student was permitted to penetrate into the arcana he was compelled to join a brotherhood and to take an oath of secrecy. Things have, of course, considerably changed since those days. The powers that be have become more and more convinced of the paramount necessity of improvement in the direction of mathematical training. And with the courage and directness which is such a marked characteristic of this astonishing race, they refuse to allow entrance to any of the higher courses at the University or higher technical institutions to those who have not the necessary preliminary mathematical equipment. We shall be very much mistaken if the results of these drastic measures do not make themselves very evident in the course of the next half-century.

The feeling for mathematics in Japan is of no new growth. In the royal library of Tokio there are no fewer than 2,000 mathematical works, some of them going back to the sixteenth century. The essential characteristic of the Japanese mind is a devotion to matters of practical value. This accounts for the lines on which their development took place. First and foremost among the practical questions which attracted their investiga-

tions was one of the three problems of antiquity, that of the "squaring of the circle," as it is generally called. To the discovery of the ratio of the circumference of the circle to its diameter they devoted considerable attention, and with a large measure of success. The history of the search for this ratio, the Ludolphian number (so named after Ludolf van Ceulen, who gave up his life to its discovery and succeeded in calculating it to 333 places of decimals) is practically the early history of mathematics itself. Even the geometrical proof that the number is incommensurable, given by the great Newton in his *Principia*, leads to a result which is not always true, so that his very condensed argument cannot be flawless. The best way of getting an approximate result is by means of infinite series, and these series are obtained by the use of the binomial theorem for fractional exponents. The astonishing thing is that this method was utilised by the Japanese, so that the theorem, in which Leibniz found to his disgust in 1676 that he had been forestalled by Sir Isaac Newton, was known about the same time to the Japanese, and must therefore have been discovered by them independently. It is also pretty certain that they also discovered independently the infinitesimal calculus, another of Newton's triumphs. But whatever may have been the cause, they appear to have achieved nothing of any importance after the seventeenth century. In Europe the progress of the higher mathematics proceeded apace. But the Far East possessed no minds of the order of a Euler or a Gauss. So the interest in the higher branches gradually waned and presently fell into abeyance altogether. The tide is beginning to turn, and the chances are that the future may have for us its surprises. Even the war did not stay the gifted few who are engaged on mathematical research. From Russia and from Tokio alike there has been no check in the regular appearance of the mathematical and physical periodicals with their papers and memoirs. A nation that once produced its Newton may yet produce another.

And what about the neighbours of Japan? It used to be supposed that the Chinese had made great progress in theoretical mathematics. This was due to a misapprehension on the part of a Jesuit missionary who did not know that the sole duty of the important department known as the Board of Mathematics was to prepare an annual almanac with its dates and predictions, and not to promote the study of mathematics throughout the Empire. But even while we are writing we hear that there is promise for the future in China. The following theorem was recently sent to a Japanese mathematician by one of his Chinese friends.

Let $ABCD$ be a convex in-quadrilateral to a circle of centre O; 2α, 2β, 2γ, 2δ the smaller arcs subtended by the chords AB, BC, CD, DA; r_a, r_b,

r_c, r_d the in-radii of the triangles BCD, CDA, DAB, ABC. Then since in any triangle we have $r=4R\Pi\sin\frac{A}{2}$, we have at once

$$r_a=4R\sin\beta\sin\gamma\sin(\alpha+\delta)\,;\quad r_c=4R\sin\alpha\sin\delta\sin(\beta+\gamma),$$

whence $$r_a+r_c=4R\Sigma\sin\alpha\sin\beta\sin\gamma\cos\delta,$$

and by symmetry $$=r_b+r_d.$$

Now consider a polygon $A_1A_2\ldots A_n$ inscribed in a circle, and let S_r be be the sum of the in-radii of the triangles which have A_r for a common vertex and the sides of the polygon (except those on which A_r lies) as sides opposite A_r.

Then shall $S_1=S_2=\ldots=S_{n-1}=S_n$.

For if $S_1=S_r$ in the polygon $A_1A_2\ldots A_r\ldots A_nA_1$, then by analogy $S_1'=S_r'$ in the polygon $A_1A_2\ldots A_r\ldots A_nA_{n+1}A_1$.

This is clearly seen, because

$$S_1'=S_1+r_1\,;\quad S_r'=S_r+r_2+r_3-r_4,\ \text{where } r_1, r_2, r_3, r_4$$

denote the in-radii of the triangles $A_1A_nA_{n+1}$, $A_rA_nA_{n+1}$, $A_rA_{n+1}A_1$, $A_rA_nA_1$. But in the quadrilateral $A_rA_nA_{n+1}A_1$, we have $r_1+r_4=r_2+r_3$; $\therefore S_1=S_r$, and it follows that $S_1'=S_r'$. (*Mathesis*, Dec., 1905.)

This resurrection of Chinese mathematics reminds us of the poet's lines:

"She let the legions thunder past
And plunged in thought again."

W. J. G.

REVIEWS.

Œuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences, par Émile Picard. (Paris: Gauthier-Villars, 1905). Tome I. Pp. xl, 498.

The appearance of this volume is very welcome, and its successors will be eagerly awaited. The editor, assisted by the late Prof. Xavier Stouff, has corrected various misprints, 'sometimes at the cost of long calculations'; this, and the addition of some brief notes, will save the reader much trouble, and occasional risk of error (*e.g.* on p. 163 there is a correction of a casual mistake in one of Hermite's letters to Jacobi).

This volume contains, amongst other things, the memoirs on quadratic forms, on homogeneous binary forms, and on the theory of the transformation of Abelian functions. Apparently the arrangement is chronological, but although the source of each paper is given, the date is not always stated, and this is somewhat of a blemish in an edition of this kind. Possibly a dated list is reserved for the final volume.

By way of preface the editor has reprinted the admirable lecture on the scientific work of Hermite which he delivered at the Sorbonne in 1901. The frontispiece is a portrait of Hermite at the age of about 25, very youthful, somewhat roguish, but with solid brow and brooding eyes. In size the book is convenient, being a large octavo, and it is as needless to commend the printing as the competency of the editor.

G. B. Mathews.

Sur les Systèmes triplement indéterminés et sur les Systèmes triple-orthogonaux. Par C. Guichard (*Scientia*, June, 1905). Paris: Gauthier-Villars, 1905. Pp. 96.

The object of this memoir is indicated in the title; the systems considered are of points, lines, and planes, and the work is a development of the author's previous investigations (*Ann. de l'Ec. N.* '97, '98, '03). The method is of the school of Darboux. The principal feature of novelty is the use made of certain orthogonal determinants connected with the system discussed. Various interesting results are obtained and problems suggested; to explain them here would take too much space, but those interested in partial differential equations or infinitesimal geometry will find this memoir very attractive. G. B. MATHEWS.

Abstrakte Geometrie. BY KARL THEODOR VAHLEN. (Teubner, Leipzig, 1905.)

This book is not a text-book but an account of the investigations of its author in the domain of the Axioms of Geometry, Euclidean and, so-called, non-Euclidean. Though it claims to be complete in itself, we venture to think that the book would be almost unintelligible to anyone who was not somewhat deeply read in the recent German and Italian literature of the subject. Moreover, published as it is in book form and not in the Proceedings of a learned Society, it has presumably escaped the criticism of referees. In any case, though it is attractively got up, and contains much suggestive matter, it cannot be regarded in its present form as presenting that "strictly logical and flawless system of axioms and postulates for each of the three possible geometries," the construction of which is stated in the preface to be the main object the author has had in view. Coming to details, the uninitiated English reader will be at once struck by the free use in the preliminary portion of the work of the ideas and nomenclature of the theory of sets of points. This, the youngest of all branches of mathematical science, is fast extending its sway over all the other branches. The next portion of the work is devoted to projective geometry, and the attempt is made to base this branch entirely on (1) the single idea of point as fundamental concept, and (2) the fact that there is at least one point A and at least one point B other than A as single fundamental axiom, thus avoiding in appearance at least the introduction, for example, of the straight line as fundamental concept. Here, as may be supposed, the object is in the main not to obtain new results, but to show how the axioms assumed lead to facts of common knowledge. The reasoning is often rather involved and though symbols are freely used the proof of a single proposition often occupies many pages. In the first instance, the axioms of order and continuity are avoided, as well as that of parallels. As examples of the results so obtained, we may cite the theorem that the necessary and sufficient condition that a plane geometry should be the section of a space geometry is that the theorem of Desargues should hold good, a result proved previously by Hibbert, but only on the assumption of the truth of the parallel axiom. The Pascal theorem is then proved by means of the axioms of order and continuity.

The remainder of the book is devoted to "Affine" geometry and metrical geometry. We should like in conclusion to put in a plea for the overworked modern mathematician. Every memoir is supposed to have prefixed to it a full account of the contents of the paper, pointing out what parts are new either in matter or mode of treatment. This useful rule is only too often transgressed, and notably so in the present instance. The only account of the subject matter, which occupies three hundred pages in all, is contained in barely two pages of preface. In an ordinary text-book a table of contents sufficiently supplements the preface, but in the case of a book such as the one before us we think much more was required. W. H. YOUNG.

Correspondance d'Hermite et de Stieltjes. (Gauthier-Villars, 1905.)

These two volumes contain 432 letters, extending from 1882 to the death of Stieltjes in 1894; the appendix consists of 4 letters from Stieltjes to Mittag-Leffler. The letters had been prepared for the press in Hermite's lifetime, and it appears that Hermite had promised to write an introduction to them; but this duty has been left to M. Picard (Hermite's son-in-law). The editors are MM. Baillaud and Bourget, colleagues of Stieltjes at the University of Toulouse; they have added a portrait and memoir of Stieltjes, two portraits of Hermite and a facsimile of Stieltjes' manuscript, as well as numerous notes scattered through the book.

The correspondence throws many interesting side-lights on the work of the two writers; it shows, for instance, how Stieltjes constantly tested theoretical results by arithmetical calculations, even at the cost of great labour. This practice was doubtless a consequence of his astronomical training in the Observatory at Leyden (1877-1883). It is noteworthy too, as Picard says in the Preface, "que sous cette forme plus personnelle le langage abstrait de l'Analyse perde de sa sécheresse et que la Mathématique y devienne plus humaine."

An adequate description of the results given in the letters is impossible here, owing to lack of space. But it may be noted that Stieltjes began his work with investigations on the Theory of Numbers, which led him to Riemann's ζ-function, and thence he passed to more general function-theory. His investigations on Continued Fractions seem to date from the beginning of 1887, so far as can be seen from the letters; this was shortly after his appointment (October, 1886) to Toulouse; and he continued working at the theory as long as strength remained.

A few elegant results, taken almost at random, may be mentioned. Letter 130 (October, 1888) contains the beautiful integral of Euler's differential equation,

$$\frac{dx}{X^{\frac{1}{2}}}=\frac{dy}{Y^{\frac{1}{2}}},\ X=(a_0,\ a_1,\ a_2,\ a_3,\ a_4 \between x,\ 1)^4,$$

namely:

$$\begin{vmatrix} a_0, & a_1, & a_2-2c, & 1 \\ a_1, & a_2+c, & a_3, & -\frac{1}{2}(x+y) \\ a_2-2c, & a_3, & a_4, & xy \\ 1, & -\frac{1}{2}(x+y), & xy, & 0 \end{vmatrix}=0$$

where c is the arbitrary constant. Letter 290 investigates Klein's theorem on the number of zeros of the hypergeometric series between 0 and 1; the method seems to be not essentially different from Hurwitz's (*Math. Annalen*, Bd. 38). In letter 393, Stieltjes shows that even if the sum of an infinite series is arbitrarily small for *all* real values of x, yet it cannot be deduced that *all* the coefficients are correspondingly small. Letter 399 contains a striking result on uniform convergence, which is reproduced in facsimile of Stieltjes' writing at the end of Vol. II.

It is curious to see that Hermite felt the burden of University Examinations. In letter 5, he says, "Ce sont les devoirs universitaires, qui surchargent les pauvres géomètres et entravent leurs recherches. A la Sorbonne nous avons maintenant une session d'examens de baccalauréat et j'ai le regret de passer bien du temps à lire des compositions et à interroger sur l'Arithmétique la Géométrie élémentaire, etc."; and again (in 247) "Je ne puis vous dire à quels efforts je suis condamné pour comprendre quelque chose aux épures de la Géométrie descriptive, que je déteste, et à des choses comme la formule des annuités en Arithmétique, etc." He envies, too, the German professors who can hand over to their pupils the work of completing the calculations encountered in their researches; and says (in 418) "Ces calculs, je les ferai moi-même, en me trompant et les recommençant, en invoquant vainement, je le crains bien, la devise de Guillaume d'Orange."

Doubtless many English workers will sympathise with the feelings of the great Frenchman. T. J. I'a. B.

Sitzungsberichte der Berliner Mathematischen Gesellschaft (Third Year, 1904). Extracted from the Archiv f. Math. u. Phys. (3), Bd. 7, 8. (Teubner.)

This volume of 85 pages contains several papers of interest, amongst the results in which the following may be mentioned:

Kneser gives an elegant proof (independently of the theory of Fourier's series) that if

$$\int_0^{2\pi} f(x)\cos mx\,dx=0,\quad \int_0^{2\pi} f(x)\sin mx\,dx=0,\quad (m=0,\,1,\,2,\,3,\,\ldots)$$

then $f(x)$ is identically zero.

Koppe has a note on the base of Neper's (Napier's) table of logarithms. It has been stated by many writers that Neper attached the logarithm $n\lambda$ to the number $(1-\lambda)^n$; λ being used here for 10^{-7}. As a matter of fact Neper's logarithm was the arithmetic mean between $n\lambda$ and $n\lambda/(1-\lambda)$. An easy calculation shows that Neper's base is effectively $(1/e)(1-\frac{1}{6}\lambda^2)$, if λ^3 is neglected; and thus Neper's tables (when freed from certain numerical errors) really give the natural logarithms to 14 decimals. It is interesting to observe that Neper's definition is, in modern notation, $\log r=\int_r^1 dx/x$; a definition which has been recently brought forward as a basis for the analytical theory.

Lampe gives a collection of solutions to harder problems on maxima and minima; No. 6 is:—To draw through a tangent to the (circular) base of a right cone, an elliptic section whose area is a maximum or minimum. This problem deserves mention, because the solutions sometimes given are incomplete (*e g.* Williamson's *Differential Calculus*, Art. 148: Frenet's *Exercises*, 216).

Schafheitlin gives a theorem on Bessel functions. It was suspected by Nielsen that two equations $J_m(x)=0$, $J_n(x)=0$ could not have a common root (other than $x=0$), supposing the difference $(m-n)$ to be an integer; it is here proved that with $m=n+3$, and with a special value for n (nearly equal to 4) the second root of J_n is the same as the first root of J_m. T. J. I'a. B.

Introduction to the Infinitesimal Calculus. By H. S. CARSLAW. Pp. 103. (Longmans, Green & Co., 1905.)

Kurze Einleitung in die Differential- und Integral Rechnung. Von IRVING FISHER. Pp. 72. (Teubner, 1904.)

These two little books are very much alike in character. Each is designed to serve as a short introduction to the Calculus sufficient for the purposes of a special class of students. Mr. Carslaw is thinking of engineering students, Dr. Fisher of students of economics, and the former naturally require a good deal more mathematics than the latter; and so, although the range of the subjects treated is much the same in the two books, Mr. Carslaw generally gives us a good deal more in the way of detailed applications. Each author cherishes a hope that his book may be useful as an introductory course even for mathematical students. In Mr Carslaw's case this hope is certainly justified. I have myself subjected his book to the best of all possible tests, actual use in teaching backward students, and find it admirably adapted for the purpose. But I am afraid there is hardly enough detail in Dr. Fisher's book to make it very useful in such cases.

A great merit of Mr. Carslaw's book is his treatment of the elements of "Conics" in the only sensible way, viz.: as an easy illustration of the processes of the Calculus. He only devotes one chapter to the Conic Sections, but a great many of their elementary properties for which there is no room in the text are introduced in admirably chosen examples. It is a very great help to have an easy book from which one can teach Analytical Geometry and the Calculus at the same time. I wish that Mr. Carslaw had seen his way to include the elements of the theory of e^x and $\log x$ as well. These functions ought only to be introduced after the notions of the Calculus have been mastered, and Mr. Carslaw might have done it all in a very few pages. He says "we assume a knowledge of the properties of the following series:—$e^x=\dots$, $a^x=\dots$, $\log(1+x)=\dots$." But the kind of person who will use this book has really no such knowledge at all: at least that is my experience—and there really is no book in which exactly what is wanted can be found. It is the treatment of the exponential and the logarithm which seems to me the least satisfactory feature in both books. Both authors

assume too much, and neither makes it very clear precisely what he is assuming. On the other hand both authors handle the troublesome (though of course not really difficult) subject of "differentials" in a lucid and unobjectionable way. G. H. HARDY.

MATHEMATICAL NOTES.

183. [P. 3. b.] The name of the author of the note on Inversion in the *Gazette*, No. 47, p. 88, was accidentally omitted. The note was by Mr. V. Ramaswami Aiyar, of Arni, India. For want of space we were compelled to omit certain applications and a further development of the theorem.

184. [I. 1.] *The third approximation to the n^{th} root of a number.*

Mr. C. S. Jackson and Mr. F. J. W. Whipple have given investigations (in the May and December numbers of the *Gazette*) for the approximation*

$$(1+x)^{\frac{1}{n}}=(1+px)/(1+qx),$$

where x is small and $p=\frac{1}{2}(1+1/n)$, $q=\frac{1}{2}(1-1/n)$.

The following method follows the lines which I have suggested (*Gazette*, October, 1904) for dealing with some of the elementary power-series.

Write $$y=(1+x)-[(1+px)/(1+qx)]^n,$$

then $$\frac{dy}{dx}=1-n(p-q)\frac{(1+px)^{n-1}}{(1+qx)^{n+1}};$$

so that $\frac{dy}{dx}$ vanishes with x, if $n(p-q)=1$.

Then $$\frac{d^2y}{dx^2}=\frac{(1+px)^{n-1}}{(1+qx)^{n+1}}\left[\frac{(n+1)q}{1+qx}-\frac{(n-1)p}{1+px}\right],$$

which also vanishes with x, if $(n-1)p=(n+1)q$.

Thus $p=\frac{1}{2}(1+1/n)$, $q=\frac{1}{2}(1-1/n)$, as given above; with these values we find

$$\frac{d^2y}{dx^2}=\frac{n^2-1}{2n^2}x\frac{(1+px)^{n-2}}{(1+qx)^{n+2}}.$$

If we differentiate $(1+px)^{n-2}(1+qx)^{-(n+2)}$, it will be found that the result is

$$-(1+px)^{n-3}(1+qx)^{-(n+3)}(1+4pqx).$$

Thus $(1+px)^{n-2}(1+qx)^{-(n+2)}$ decreases as x increases from 0; and consequently is less than 1 for positive values of x.

That is, $$\frac{d^2y}{dx^2}<\frac{n^2-1}{2n^2}x \quad\text{or}\quad \frac{d}{dx}\left(\frac{dy}{dx}-\frac{n^2-1}{4n^2}x^2\right)<0,$$ x being positive

Now $\frac{dy}{dx}-\frac{n^2-1}{4n^2}x^2$ is zero for $x=0$, and so is negative for positive values of x.

That is, $$\frac{d}{dx}\left(y-\frac{n^2-1}{12n^2}x^3\right)<0,$$

and by the same argument,

$$y-\frac{n^2-1}{12n^2}x^3<0.$$

Similarly, since $\frac{d^2y}{dx^2}$ is positive for positive values of x, we infer that y is positive; so that

$$0<y<\tfrac{1}{12}(1-1/n^2)x^3, \text{ if } x \text{ is positive.}$$

---

* To reduce this to the form given, write $x=N/a^n$, and multiply both sides by a.

Since $(1+px)/(1+qx)$ is greater than 1, it now follows that *

$$0<(1+x)^{\frac{1}{n}}-(1+px)/(1+qx)<y/n\ ;$$

so that *the error in the approximation (for positive values of x) is less than*

$$\tfrac{1}{12}(n^2-1)x^3/n^3,$$

provided n has any rational value greater than 1.

Mr. Jackson gave this result as an approximation to the error, but without noticing that the estimated error was really in excess of the true error, for positive values of x.

In the special cases $n=2$, 3 (which are those of chief interest, perhaps), it is easy to verify by direct multiplication that

$$(1+x)(1+\tfrac{1}{4}x)^2-(1+\tfrac{3}{4}x)^2=\tfrac{1}{16}x^3<\tfrac{1}{16}x^3(1+\tfrac{1}{4}x)^2,$$

$$(1+x)(1+\tfrac{1}{3}x)^3-(1+\tfrac{2}{3}x)^3=\tfrac{1}{27}(2x^3+x^4)<\tfrac{2}{27}x^3(1+\tfrac{1}{3}x)^3,$$

leading to

$$(1+x)^{\frac{1}{2}}-(1+\tfrac{3}{4}x)/(1+\tfrac{1}{4}x)<\tfrac{1}{32}x^3,$$

$$(1+x)^{\frac{1}{3}}-(1+\tfrac{2}{3}x)/(1+\tfrac{1}{3}x)<\tfrac{2}{81}x^3.$$

These are the same results as were found above by the aid of the calculus; but to extend the algebraic method to any integral value of n seems rather laborious, although some of the transformations are interesting.

It may be worth while to recall the fact that Dedekind makes use of this approximation (for $n=2$) in his theory of irrational numbers.

T. J. I'a. Bromwich.

185. [A. 1. c.] *Note on a point in the demonstration of the Binomial Theorem.*

When x is numerically less than 1, the series $1+mx+\dfrac{m(m-1)}{1\,.\,2}x^2+\ldots$ is absolutely convergent for all values of m, and denoting it by $f(m)$ we establish the addition theorem that $f(m)\,.\,f(n)=f(m+n)$.

The Binomial Theorem is, that $f(m)$ is, for all values of m, equal to $(1+x)^m$; that is, to the *positive* quantity which, under the theory of Indices, is represented by the expression $(1+x)^m$.

Usually, the theorem is first shown to be true for positive integral values of m, and then it is considered for positive fractional values, and finally for negative values of m.

In dealing with the case when m is a positive fraction $=\dfrac{p}{q}$, what is usually done is, after proving that the qth power of $f\left(\dfrac{p}{q}\right)$ is equal to $(1+x)^p$, it is quietly assumed that $f\left(\dfrac{p}{q}\right)=(1+x)^{\frac{p}{q}}$. But if q be an even integer, might not $f\left(\dfrac{p}{q}\right)$ be rather equal to $-(1+x)^{\frac{p}{q}}$?

This difficulty (so far as I know) is raised and discussed only in *Chrystal*; but the discussion there is by no means elementary. The point, however, admits of an elementnry treatment, by taking the cases as follows.

Case I. If m is a positive integer, we have $f(m)=\{f(1)\}^m=(1+x)^m$, and there is no ambiguity.

Case II. If m is a negative integer, $=-n$ say, we have

$$f(-n)\,.\,f(n)=f(0)=1\ ;$$

---

* If a, b, n are all greater than 1, $a^n-b^n>n(a-b)$. This is obvious on division if n is an integer; if n is not an integer we have

$$\frac{d}{dx}(x^n-1)=nx^{n-1}>n$$

if $x>1$. Thus $x^n-1>n(x-1)$. Write $x=a/b$, and the result follows.

$$\therefore\ f(m)=f(-n)$$
$$=\frac{1}{f(n)}=\frac{1}{(1+x)^n},\text{ by Case I.,}$$
$$=(1+x)^{-n},\text{ by the theory of Indices,}$$
$$=(1+x)^m.$$

CASE III. Let m be a negative quantity numerically less than unity, $=-m'$ say.

Then
$$f(m)=1+mx+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3+\dots$$
$$=1-m'x+\frac{m'(m'+1)}{1.2}x^2-\frac{m'(m'+1)(m'+2)}{1.2.3}x^3+\dots.$$

If x be negative, each term in the series is positive, and therefore $f(m)$ is positive. If x be positive, the terms are alternately positive and negative, and each term is numerically greater than the preceding term. Hence in this case also the value of $f(m)$ is positive. Thus, whether x be positive or negative, $f(m)$ is, *a priori*, seen to be a positive quantity.

Let now $m=-p/q$ where p and q are positive integers, $p<q$.

Then
$$\{f(m)\}^q=f(mq)=f(-p)$$
$$=(1+x)^{-p},\text{ by Case II.};$$

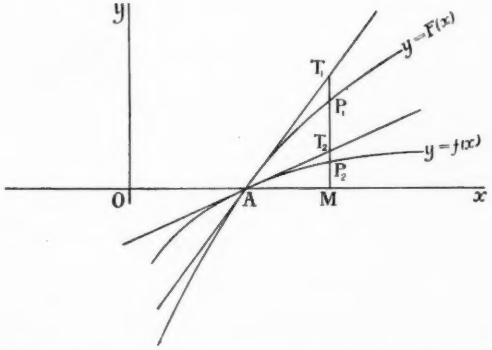
$\therefore\ f(m)=(1+x)^{-\frac{p}{q}}$, that is, the *positive* value denoted by this expression,
$$=(1+x)^m.$$

CASE IV. Lastly, let m be a quantity not falling under any of the above cases. Then we can put $m=n-m'$ where n is a positive or negative integer, and m' is positive and less than 1. Accordingly,
$$f(m)=f(n-m')=f(n).f(-m')$$
$$=(1+x)^n.(1+x)^{-m'},\text{ by Cases I. or II., and III.,}$$
$$=(1+x)^{n-m'},\text{ by the theory of Indices,}$$
$$=(1+x)^m.$$

V. RAMASWAMI AIYAR.

186. [C. 1. e. α.] *Indeterminate Forms.*

A geometrical illustration.



Let $y=F(x)$ and $y=f(x)$ be the equations of two curves, each intersecting the axis of x at the point $x=a$.

The ratio of corresponding ordinates is indeterminate when $x=a$.
But, drawing the tangents $AT_1 AT_2$ at A, we have

$$\frac{F(x)}{f(x)}=\frac{P_1M}{P_2M}=\frac{P_1M}{T_1M}\,\frac{T_1M}{T_2M}\,\frac{T_2M}{P_2M}$$

$$=\frac{P_1M}{T_1M}\cdot\frac{AM\tan T_1AM}{AM\tan T_2AM}\cdot\frac{T_2M}{P_2M},$$

and therefore when AM diminishes without limit

$$\underset{x=a}{\mathrm{Lt}}\frac{F(x)}{f(x)}=1\times\frac{\tan T_1AM}{\tan T_2AM}\times 1$$

$$=\frac{F'(a)}{f(a)}.$$

The extension to the case where $F'(a)$ and $f'(a)$ both vanish is obvious.

C. S. JACKSON.

187. [J. 5.] *The Continuum.*

Readers of the *Gazette* may be interested to know that Professor E. V. Huntington has concluded his series of articles in the *Annals*, "Mathematics on the Continuum" and the "Transfinite Numbers." They form an *elementary* introduction to some of the problems so actively debated at the present time in the field of Cantor's *Mengenlehre*. W. J. G.

188. [C. 1. e.] *Proof of Taylor's Theorem.*

Let $$R=f(z)-f(z-h)-hf'(z-h)-\frac{h^2}{\underline{|2}}f''(z-h)-\ldots-\frac{h^n}{\underline{|n}}f^{(n)}(z-h),$$

then $$\frac{dR}{dh}=\frac{h^n}{\underline{|n}}f^{(n+1)}(z-h).$$

Keeping z constant, $$R=\int\frac{h^n}{\underline{|n}}f^{(n+1)}(z-h)\,dh+\text{const.}$$

But $R=0$, if $h=0$;

$$\therefore\ R=\int_0^h\frac{h^n}{\underline{|n}}f^{(n+1)}(z-h)\,dh,$$

which can be transformed or discussed in the usual way. The only thing the student has to remember is to put z for $x+h$, and therefore $z-h$ for x in the usual formula. G. H. BRYAN.

189. [K. 13. a.] The *Remarque Minuscule* (Note 167, *Gazette*, May 1905) occurred to me also in 1887, and has been set in Aberystwyth and University of Wales' Examinations. It is, however, probably older, and contained in Bellavite's striking theorem, viz. if $ABCD\ldots$, $A'B'C'D'\ldots$ be similar polygons inversely situated, and if $AA'BB'CC'$, etc., be divided at P, Q, R, etc., each in the ratio of the linear dimensions of the polygons, then P, Q, R, etc., lie in a straight line.

The following extension of simple proof by Vector methods. If P_1, P_2 be points in AA' such that $AP_1 . AP_2 : A'P_1 . A'P_2 :: AB^2 : A'B'^2$, and if BB' be divided at Q_1Q_2 similarly to AP_1P_2A', then P_1Q_1 P_2Q_2 are equally inclined to AB, $A'B'$. R. W. GENESE.

190. [V. 1. a.]. *An apparatus for teaching long multiplication.*

I devised this, in the first instance, to enable my pupils to change from the old method (left to right) to the new (right to left), without confusion.

It consists essentially of a blackboard composed of four (or more) rectangular slats ($1{\cdot}2\times{\cdot}15$ m.'s each) which slide in horizontal grooves. The local carpenter made mine for 12s. 6d.

The following examples will show the use of this device :

Name the slats, from the top one downwards, A, B, C, D.

On A write any number, say 2763.

Multiply by 7 ; write the result, 19341, on B so that the 4 is directly beneath A's 6.

(1) Add. Verify, result is $A \times 8$. Note that (1) is "useless."

(2) Move slat B one place to the left. Add. Verify, result is $A \times 71$. Remove B. Multiply A by 70, adding in A concurrently. Thus, for 2763×71 we have :

"3". Write 3.
"$7 \times 3 = 21$, and 6 make 27." Write 7.
"$7 \times 6 = 42$, carry 2 ; 44, and 7 make 51." Write 1.
"$7 \times 7 = 49$, carry 5 ; 54, and 2 make 56." Write 6.
"$7 \times 2 = 14$, carry 5 ; 19." Write 19.

Answer, 196173. Verify,

$$\begin{array}{r} 2763\times 71 \\ 19341 \\ \hline 196173 \end{array}$$

(3) Replace B ; move it one place to the right. Add. Verify, result is $A \times 17$. Remove B. Multiply A by 7, adding in $10A$ concurrently. Thus, for 2763×17 we have :

"$7 \times 3 = 21$." Write 1.
"$7 \times 6 = 42$, carry 2 ; 44, and 3 make 47." Write 7.
"$7 \times 7 = 49$, carry 4 ; 53, and 6 make 59." Write 9.
"$7 \times 2 = 14$, carry 5 ; 19, and 7 make 26." Write 6.
"Carry 2 ; 2, and 2 make 4." Write 4.

Answer, 46971. Verify,

$$\begin{array}{l} 2763 \;\times 17 \\ 19341 \\ \hline 46971 \end{array}$$

(4) Replace B. Extend (2) and (3) for multiplication by 701, 107, etc.

(5) Replace B. Multiply A by 2 ; write the result, 5526, on C so that the 6 is directly below A's 3.

Add. Verify, result is $A \times 10$ (c).

Place slats successively in position for multiplication by 27, 72, 271, 721, 172, 127, 217, 712, 19, 91, 73, 37, 28, 82.

E.g.

$$\begin{array}{r} 2763\times 72\ (a) \\ 19341 \\ 5526 \\ \hline 198936 \end{array} \qquad \begin{array}{r} 2763\times 721\ (a) \\ 19341 \\ 5526 \\ \hline 1992123 \end{array} \qquad \begin{array}{l} 2763\times 127\ (b) \\ 19341 \\ 5526 \\ \hline 350901 \end{array}$$

$$\begin{array}{r} 2763\times 712\ (a) \\ 19341 \\ 5526 \\ \hline 1967256 \end{array} \qquad \begin{array}{l} 2763\times 19\ (c) \\ 19341 \\ \;\;5526 \\ \hline 52497 \end{array} \qquad \begin{array}{l} 2763\times 28\ (c) \\ 19341 \\ 5526 \\ \hline 77364 \end{array}$$

Note that (a) are correct "form" and "useful," (b) is incorrect "form"—correct by interchanging B and C—and "useful," while (c) are "useless."

(6) Extend (5) for multiplication by 7021, etc., etc.

(7) Remove C. Move B one place to the left. Subtract A.

Thus

$$\begin{array}{r} 2763\times(70-1) \\ 19341 \\ \hline 190647 \end{array}$$

Remove *B*. Multiply *A* by 70, subtracting *A* concurrently.

Thus, for $2763 \times (70-1)$ we have:

"3 from 10 leave 7. Carry minus 1." Write 7.

"$7 \times 3 = 21$, minus 1; 20. 6 from 20 leave 14." Write 4.

"$7 \times 6 = 42$, carry 1; 43. 7 from 43 leave 36." Write 6

"$7 \times 7 = 49$, carry 3; 52. 2 from 52 leave 50." Write 0.

"$7 \times 2 = 14$, carry 5; 19." Write 19.

Answer, 190647. Verify,

$$\begin{array}{r} 2763 \times 69 \\ \hline 16578 \\ 24867 \\ \hline 190647 \end{array}$$

Note that (7) is harder than (2)-(6). Dull boys fail to grasp it, though they easily comprehend the rest of the system.

The board is also useful for the teaching of approximations and contracted (or Italian) long division.

F. J. O. CODDINGTON.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) **For Sale.**

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and published by E. HENDRICKS, M.A., Des Moines, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

Cayley's Mathematical Works. Complete, equal to new, £10. Apply, Professor of Mathematics, University College, Bangor.

The Mathematical Gazette. Nos. 7-18 inclusive, £1. No. 8 is out of print and extremely scarce.

Sectionum Conicarum. Lib. Sept. A. ROBERTSON. Oxonii. MDCCXCII. (376 pp. quarto.) 3s. 6d.

(2) **Wanted.**

The Messenger of Mathematics. Vols. 24, 25.

Tortolini's Annali. Vol. I. (1850), or any of the first eight parts of the volume.

Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

Mathematical Questions and Solutions from the Educational Times. Vol. 18.

Cayley's Collected Mathematical Papers. Vols. VII.-XIII.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.

gh

ted
ON.

ca.
ND-

cal

he

ly,

nt

II.

he

in
ve

8.